## CALCULUS.

### NOTE ON PROBLEM 84, BY DR. E. WOELFFING, STUTTGART, GERMANY.

The solution of question 84 can be found in some theorems proved by W. Merkelbach (*Ueber Rollkurven welche von einer Graden eingehüllt werden*, Diss. Marburg, 1881). The first of them is the following:

If a curve, $C$, rolls upon another curve, $C'$, and a point, $P$, in the plane of $C$ describes a straight line $L$, and we make afterwards the curve $C'$ roll upon the curve $C$, then a straight line, $L$, in the plane of $C'$ will always pass through $P$ (page 18 of the paper quoted).

Now (and this is the second theorem of Merkelbach) if a sinusoid rolls upon an ellipse, a straight line in the plane of the former passes through a focus of the latter (page 24); therefore, if the ellipse rolls upon the sinusoid, any one of the foci of the former will describe a straight line.

*Stuttgart, Germany, July 19, 1899.*

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91. Proposed by GUY B. COLLIER, Schenectady, N. Y.

Find the area of a loop of the curve $r^2\cos\theta = a^2\sin3\theta$. [From Hall's *Differential and Integral Calculus*].

I. Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; M. C. STEVENS, A. M., Professor of Mathematics, Purdue University, Lafayette, Ind.; WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.; ELMER SCHUYLER, Reading, Penna.; and J. SCHEFFER, A. M., Hagerstown, Md.

The curve has two equal loops, one in the first and the other in the third quadrant.

The limits of $\theta$ are 0 and $\frac{1}{3}\pi$.

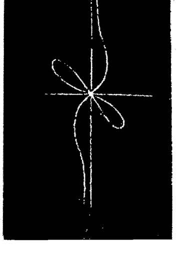
$$\therefore\ A = \tfrac{1}{2}\int_0^{\frac{1}{3}\pi} r^2 d\theta = \tfrac{1}{2}a^2\int_0^{\frac{1}{3}\pi} \frac{\sin3\theta d\theta}{\cos\theta}$$

$$= \tfrac{1}{2}a^2\int_0^{\frac{1}{3}\pi} (4\sin\theta\cos\theta - \tan\theta)d\theta = \tfrac{1}{4}a^2(3-2\log2).$$

II. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; and GEORGE LILLEY, Ph. D., Professor of Mathematics, University of Oregon, Eugene, Ore.

The shape of the curve is seen in the diagram.

The limits are evidently from 0° to 30° for a loop.



$$\text{Then } A = \tfrac{1}{2}\int r^2 d\theta = \tfrac{1}{2}\int_0^{\frac{1}{6}\pi} \frac{a^2\sin3\theta}{\cos\theta} d\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{1}{6}\pi} \frac{3\sin\theta - 4\sin^3\theta}{\cos\theta} d\theta = \frac{a^2}{2}\int_0^{\frac{1}{6}\pi} 3\tan\theta d\theta$$

$$-2a^2\int_0^{\frac{1}{6}\pi} (\tan\theta - \sin\theta\cos\theta)d\theta$$

$$=\frac{3a^2}{2}\log\sec\theta-2a^2\log\sec\theta+a^2\sin^2\theta\Big]_0^{\frac{1}{6}\pi}=a^2\sin^2\theta-\frac{a^2}{2}\log\sec\theta\Big]_0^{\frac{1}{6}\pi}$$

$$=\frac{3a^2}{4}-\frac{a^2}{2}\log 2.$$

[NOTE. In the figure, the loop in the fourth quadrant should be in the first, and the one in the second in the third.

**92. Proposed by B. F. SINE, Principal of Capon Bridge Normal School, Capon Bridge, W. Va.**

How much wood is taken from a log 12 inches in diameter, by boring a two-inch hole through the center, the axis of the hole being perpendicular to axis of log ?

**I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; J SCHEFFER, A. M., Hagerstown, Md., and WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.**

$V=8\int\int\int dxdydz=$volume.

The equation of the surface of the cylinder corresponding to the log is

$$x^2+y^2=36=R^2,$$

and the equation of the surface of the cylinder corresponding to the auger-hole is

$$x^2+z^2=1=r^2.$$

$$V=8\int\int\int ydxdz=8\int_o^r\int_0^{\sqrt{(r^2-x^2)}}\sqrt{(R^2-x^2)}dxdy.$$

$$\therefore\ V=8\int_o^r\sqrt{[(r^2-x^2)(R^2-x^2)]}dx$$

$$=\tfrac{8}{3}R^3\{[1+(r/R)^2]E(r/R)-[1-(r/R)^2]F(r/R)\}$$

$$=576\{(1+\tfrac{1}{36})E(\tfrac{1}{6})-(1-\tfrac{1}{36})F(\tfrac{1}{6})\}$$

$$=576\{\tfrac{37}{36}E(\tfrac{1}{6})-\tfrac{35}{36}F(\tfrac{1}{6})\}\text{ cubic inches.}$$

Also $V=8\int_o^r\sqrt{(r^2-x^2)}\left[R-\frac{x^2}{2R}-\frac{x^4}{8R^3}-\dots-\frac{1.3.5.7\dots(2n-1)x^{2n}}{2.4.6\dots 2nR^{2n-1}}\right]dx.$

Now $\int_o^r\sqrt{(r^2-x^2)}x^{2n}dx=\frac{1.3.5.7\dots(2n-1)}{2.4.6.8\dots(2n+2)}\pi r^{n+2}.$

$$\therefore\ V=4r^2R\pi\left[\tfrac{1}{2}-\frac{r^2}{16R^2}-\frac{r^4}{128R^4}-\dots-\frac{(1.3.5.7\dots 2n-3)^2(2n-1)r^{2n}}{(2.4.6\dots\dots 2n)^2(2n+2)R^{2n}}-\dots\right]$$

$\therefore\ V=24\pi(\tfrac{1}{2}-\tfrac{1}{576}-\tfrac{1}{165888}-\tfrac{5}{95551488}-\dots-)=37.56784$ cubic inches.